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Abstract. We obtain upper bounds to spin correlation functions in the thermodynamic limit of the zero external field limit for the Ising model of general spins and for the n-vector model assuming that the exchange integrals are quenched random variables and that their probability distributions are continuous. The upper bounds involve, as a factor, the corresponding spin correlation function of a uniform ferromagnetic Ising model of spin ± 1, in which the exchange integral is determined by the distributions of the random exchange integrals of the original system. By using these upper bounds, we find a sufficient condition on the probability distributions of the exchange integrals for the disappearance of the ferromagnetic and antiferromagnetic states in the system.

1. Introduction

In a previous paper (Horiguchi and Morita 1981), we obtained upper bounds to the spin correlation functions in the thermodynamic limit of the zero external field limit for random-bond Ising models and also for a random-bond n-vector model, where the exchange integrals are assumed to take either the values $J > 0$ and $-J$ with probabilities p and $1-p$, respectively, or the values J, 0 and $-J$ with probabilities p, r and $1-p-r$, respectively. By applying these upper bounds to the spontaneous magnetisation, we proved that the systems cannot have spontaneous magnetisation in a range of concentration of the ferromagnetic bonds.

In this paper, we extend previous studies to the systems described by the random-bond Ising model or by the random-bond n-vector model in which probability distributions of the exchange integrals are continuous. We give a sufficient condition on the probability distributions of the exchange integrals for the disappearance of the ferromagnetic and antiferromagnetic states in the system. This condition is investigated for several special types of the probability distribution, in detail. The nature of the systems in which the probability distribution of the nearest-neighbour exchange integral is the Gaussian distribution has been studied extensively by many authors (Edwards and Anderson 1975, Sherrington and Kirkpatrick 1975, Klein *et al* 1979 and so on). We show exactly that the system is not in the ferromagnetic nor in the antiferromagnetic state, e.g. for $|\bar{J}/\sigma| < 0.49344$ on the square lattice and for $|\bar{J}/\sigma| < 0.26826$ on the simple cubic lattice, etc, where $\bar{J}$ is the mean of the nearest-neighbour exchange integral and σ its standard deviation.

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In § 2, we prove an inequality of the spin correlation function for the random-bond Ising model with general spin S. The random-bond n-vector model is considered in § 3. The disappearance of the long-range order is studied for several kinds of probability distribution of the exchange integral in § 4. Concluding remarks are given in § 5.

2. Random-bond Ising model of general spin

We consider the Ising model with general spin S on a finite set Λ of N lattice sites. The total number of the sites in the set Λ is denoted as $|\Lambda|$, and hence $N=|\Lambda|$. The system is assumed to be described by the Hamiltonian

$$H=-\sum_{(i,j)} J_{ij}s_is_j-h\sum_i \mu_is_i \tag{2.1}$$

where s_i is the spin variable for the site i and takes values $-S, -S+1, \ldots, S$. For the pair of sites (i, j), J_{ij} is the exchange integral which is a quenched random variable and whose probability distribution is denoted by $\tilde{P}_{ij}(J_{ij})$. J_{ij} is independent of J_{kl} for the other pairs of sites (k, l). In the present paper, we restrict ourselves to the cases where $\tilde{P}_{ij}(-|J_{ij}|)$ is zero or non-zero according to whether $\tilde{P}_{ij}(|J_{ij}|)$ is zero or non-zero, and vice versa. This restriction excludes e.g. the rectangular distribution in the interval $[-a, b]$ except when $a=b$, although this case with $a\neq b$ has been investigated by Katsura (1977). Such a case will be studied in a separate paper by the present authors. h is the external field and μ_i is the magnetic moment of the spin on the site i. $\Sigma_{(i,j)}$ denotes the summation over all the pairs of sites belonging to Λ and Σ_i over all the sites belonging to Λ, if no restriction is stated. $\Pi_{(i,j)}$ and Π_i must be understood similarly.

For a finite set A of sites in the set Λ, the product of the spin variables for the sites in the set A is denoted by s_A^ρ:

$$s_A^\rho=\prod_{\substack{k\\(k\in A)}} s_k^{\rho(k)} \tag{2.2}$$

where $\rho(k)$ is a positive integer not greater than $2S$. We define $|A|^*$ by $\Sigma_{k\in A}\,\rho(k)$. For $S=\frac{1}{2}$, we have $\rho(k)=1$ and then $|A|^*=|A|$, that is equal to the number of sites in the set A. The canonical average of s_A^ρ is defined by

$$\langle s_A^\rho\rangle_{N,h,B_0}^{\{\beta J_{ij}\}}=\operatorname{Tr} s_A^\rho\,\mathrm{e}^{-\beta H}/\operatorname{Tr}\mathrm{e}^{-\beta H} \tag{2.3}$$

where $\beta=1/k_BT$, T is the absolute temperature and k_B is the Boltzmann constant. The subscript B_0 shows that the system is isolated; that is to say, B_0 expresses the boundary condition that the boundary spins are not coupled with the outer system even if it exists. The same average for the system of spin ±1 is denoted by $\langle\sigma_A^\rho\rangle_{N,h,B_0}^{\{\beta J_{ij}\}}$. When all the products $h\mu_i$ are replaced by their respective absolute values $|h\mu_i|$, the averages of s_A^ρ and σ_A^ρ are denoted by $\langle s_A^\rho\rangle_{N,h,B_0}^{\{\beta J_{ij}\}(+)}$ and $\langle\sigma_A^\rho\rangle_{N,h,B_0}^{\{\beta J_{ij}\}(+)}$, respectively.

In the following, we encounter the system of the Ising model of spin ±1 on a finite set Λ_1 of N_1 lattice sites, where $N_1=|\Lambda_1|$ and $A\subset\Lambda_1\subset\Lambda$. The Hamiltonian of this system is assumed to be given by

$$H_1=-\sum_{\substack{(i,j)\\(i,j\in\Lambda_1)}} J_{ij}\sigma_i\sigma_j-\tilde{h}\tilde{\mu}\sum_{\substack{i\\(i\in\Lambda_1)}}\sigma_i-\sum_{\substack{(i,j)\\(i\in\Lambda_1,j\in\Lambda\setminus\Lambda_1)}} J_{ij}\sigma_i \tag{2.4}$$

where σ_i denotes the spin variable for the site i, and $\tilde{h}$ and $\tilde{\mu}$ are positive. Later we need the following spin correlation for this system

$$\langle\sigma_A^\rho\rangle_{N_1,h,B_1}^{\{\beta J_{ij}\}} = \sum_{\{\sigma_i\}} \sigma_A^\rho\, \mathrm{e}^{-\beta H_1} \Big/ \sum_{\{\sigma_i\}} \mathrm{e}^{-\beta H_1} \tag{2.5}$$

where

$$\sigma_A^\rho = \prod_{\substack{k\\(k\in A)}} \sigma_k^{\rho(k)}. \tag{2.6}$$

B_1 expresses the boundary condition that the spins which belong to $\Lambda\backslash\Lambda_1$, and interact with a spin σ_i for $i\in\Lambda_1$, are all plus one.

The configurational average of a function $Q\{J_{ij}\}$ of the set $\{J_{ij}\}$ such as $\langle s_A^\rho\rangle_{N,h,B_0}^{\{\beta J_{ij}\}}$ is denoted by the angular brackets with suffix c

$$\langle Q\{J_{ij}\}\rangle_c = \int \ldots \int P\{J_{ij}\}Q\{J_{ij}\} \prod_{(i,j)} \mathrm{d}J_{ij} \tag{2.7}$$

where

$$P\{J_{ij}\} = \prod_{(i,j)} \tilde{P}_{ij}(J_{ij}). \tag{2.8}$$

The product $\Pi_{(i,j)}$ is taken over all the pairs of sites i and j which appear in the Hamiltonian (2.1) or (2.4). We define the thermodynamic limit of the zero external field limit as follows:

$$|\langle\langle s_A^\rho\rangle^{\{\beta J_{ij}\}}\rangle_c| \equiv \lim_{h\to+0}\lim_{N\to\infty} |\langle\langle s_A^\rho\rangle_{N,h,B_0}^{\{\beta J_{ij}\}}\rangle_c|. \tag{2.9}$$

For the set $\{\gamma_{ij}\}$ of the fixed values γ_{ij}, we define

$$\begin{aligned}\langle\sigma_A^\rho\rangle_{(+)}^{\{\gamma_{ij}\}} &= \lim_{h\to+0}\lim_{N\to\infty} \langle\sigma_A^\rho\rangle_{N,h,B_0}^{\{\gamma_{ij}\}(+)}\\ \langle s_A^\rho\rangle_{(+)}^{\{\gamma_{ij}\}} &= \lim_{h\to+0}\lim_{N\to\infty} \langle s_A^\rho\rangle_{N,h,B_0}^{\{\gamma_{ij}\}(+)}\end{aligned} \tag{2.10}$$

and at zero temperature, $T=0$ (Horiguchi and Morita 1981),

$$|\langle\langle s_A^\rho\rangle^{\{\beta J_{ij}=\infty\}}\rangle_c| = \lim_{\beta h\to+0}\lim_{N\to\infty}\lim_{\beta J_{ij}\to\infty} |\langle\langle s_A^\rho\rangle_{N,h,B_0}^{\{\beta J_{ij}\}}\rangle_c|. \tag{2.11}$$

Now we have the following theorem 1, where J_{ij}^M for each pair (i,j) is an upper bound of $|J_{ij}|$ for which $\tilde{P}_{ij}(J_{ij})+\tilde{P}_{ij}(-J_{ij})$ takes a non-zero value, if it exists, and is put equal to infinity if it does not.

Theorem 1. For β which is either finite or infinite,

$$|\langle\langle s_A^\rho\rangle^{\{\beta J_{ij}\}}\rangle_c| \leq \langle\sigma_A^\rho\rangle_{(+)}^{\{\langle|\beta_{ij}J_{ij}|\rangle_c\}}\langle s_A^\rho\rangle_{(+)}^{\{\beta J_{ij}^M\}} \tag{2.12}$$

where

$$\langle|\beta_{ij}J_{ij}|\rangle_c = \tfrac{1}{2}\int_{-\infty}^{\infty} |\ln[\tilde{P}_{ij}(J_{ij})/\tilde{P}_{ij}(-J_{ij})]|\tilde{P}_{ij}(J_{ij})\,\mathrm{d}J_{ij}. \tag{2.13}$$

Proof. We consider a subset Λ_1 of Λ which contains the finite set A: $A\subset\Lambda_1\subset\Lambda$, $|\Lambda_1|=N_1$. We define an auxiliary Hamiltonian $\tilde{H}$ by

$$\tilde{H} = -\sum_{(i,j)} J_{ij}s_is_j - h\sum_{\substack{i\\(i\in\Lambda\backslash\Lambda_1)}} \mu_is_i. \tag{2.14}$$

The canonical average of the spin variable s_A^ρ with the Hamiltonian $\tilde{H}$ is denoted by $\langle s_A^\rho\rangle_{N,(h,0),B_0}^{\{\beta J_{ij}\}}$. Then we have

$$\langle\langle s_A^\rho\rangle_{N,(h,0),B_0}^{\{\beta J_{ij}\}}\rangle_c=\int\ldots\int\Big(\prod_{(i,j)}\mathrm{d}J_{ij}\Big)\Big(\prod_{(i,j)}\alpha_{ij}\Big)\exp\Big(\sum_{(i,j)}\beta_{ij}J_{ij}\Big)\langle s_A^\rho\rangle_{N,(h,0),B_0}^{\{\beta J_{ij}\}} \tag{2.15}$$

$$\alpha_{ij}\equiv\alpha_{ij}(|J_{ij}|)=[\tilde{P}_{ij}(J_{ij})\tilde{P}_{ij}(-J_{ij})]^{1/2} \tag{2.16}$$

$$\beta_{ij}\equiv\beta_{ij}(|J_{ij}|)=\begin{cases}(1/2|J_{ij}|)\ln[\tilde{P}_{ij}(|J_{ij}|)/\tilde{P}_{ij}(-|J_{ij}|)] & J_{ij}\neq0\\ 0 & J_{ij}=0.\end{cases} \tag{2.17}$$

Equation (2.15) is now expressed as

$$\langle\langle s_A^\rho\rangle_{N,(h,0),B_0}^{\{\beta J_{ij}\}}\rangle_c=\int\ldots\int\Big(\prod_{(i,j)}\mathrm{d}J_{ij}\Big)\Big(\prod_{(i,j)}\alpha_{ij}\Big)\exp\Big(\sum_{(i,j)}\beta_{ij}J_{ij}\sigma_i\sigma_j\Big)\sigma_A^\rho\langle s_A^\rho\rangle_{N,(h,0),B_0}^{\{\beta J_{ij}\}} \tag{2.18}$$

where σ_i is either $+1$ or -1 for $i\in\Lambda_1$ and $+1$ for $i\in\Lambda\backslash\Lambda_1$. We multiply $\exp(\tilde{\beta}\tilde{h}\tilde{\mu}\sum_{i(i\in\Lambda_1)}\sigma_i)$ on both sides of (2.18), where $\tilde{\beta}$, $\tilde{h}$ and $\tilde{\mu}$ are all positive. We then take the summation with respect to $\{\sigma_i\}$ over all the possible 2^{N_1} sets of values of $\{\sigma_i\}$ and divide by $[2\cosh(\tilde{\beta}\tilde{h}\tilde{\mu})]^{N_1}$, and we have

$$\langle\langle s_A^\rho\rangle_{N,(h,0),B_0}^{\{\beta J_{ij}\}}\rangle_c=\int\ldots\int\Big(\prod_{(i,j)}\mathrm{d}J_{ij}\Big)P\{J_{ij}\}\langle\sigma_A^\rho\rangle_{N_1,\tilde{h},B_1}^{\{\beta_{ij}J_{ij}\}}\langle s_A^\rho\rangle_{N,(h,0),B_0}^{\{\beta J_{ij}\}} \tag{2.19}$$

where

$$P\{J_{ij}\}=\Big(\prod_{(i,j)}\alpha_{ij}\Big)\Big[\sum_{\{\sigma_i\}}\exp\Big(\sum_{(i,j)}\beta_{ij}J_{ij}\sigma_i\sigma_j+\tilde{\beta}\tilde{h}\tilde{\mu}\sum_{i\in\Lambda_1}\sigma_i\Big)\Big]\{[2\cosh(\tilde{\beta}\tilde{h}\tilde{\mu})]^{N_1}\}^{-1}. \tag{2.20}$$

$\langle\sigma_A^\rho\rangle_{N,\tilde{h},B_1}^{\{\beta_{ij}J_{ij}\}}$ is the canonical average at the temperature $1/k_B\tilde{\beta}$ of σ_A^ρ in the system which is composed of N_1 Ising spins of spin ±1 on the sites belonging to the set Λ_1, where the exchange integrals are $\beta_{ij}J_{ij}/\tilde{\beta}$, the external field $\tilde{h}$ and the magnetic moment $\tilde{\mu}$; see (2.4)–(2.6).

By using theorems 1 and 2 given by Horiguchi and Morita (1979) and Griffiths's inequality (Griffiths 1977), we have

$$|\langle\langle s_A^\rho\rangle_{N,(h,0),B_0}^{\{\beta J_{ij}\}}\rangle_c|\leqslant\langle|\langle\sigma_A^\rho\rangle_{N_1,\tilde{h},B_1}^{\{\beta_{ij}J_{ij}\}}|\rangle_c\langle s_A^\rho\rangle_{N,(h,0),B_0}^{\{\beta J_{ij}^M\}(+)}\leqslant\langle\sigma_A^\rho\rangle_{N_1,\tilde{h},B_1}^{\{\langle|\beta_{ij}J_{ij}|\rangle_c\}}\langle s_A^\rho\rangle_{N,(h,0),B_0}^{\{\beta J_{ij}^M\}(+)} \tag{2.21}$$

where $\langle|\beta_{ij}J_{ij}|\rangle_c$ is defined by (2.13). For the thermodynamic limit, we take the limit as $N\to\infty$ first and then as $h\to+0$. From lemma 3 given by Horiguchi and Morita (1981), we have

$$|\langle\langle s_A^\rho\rangle^{\{\beta J_{ij}\}}\rangle_c|\leqslant\langle\sigma_A^\rho\rangle_{N_1,\tilde{h},B_1}^{\{\langle|\beta_{ij}J_{ij}|\rangle_c\}}\langle s_A^\rho\rangle_{(+)}^{\{\beta J_{ij}^M\}} \tag{2.22}$$

for arbitrary N_1 and $\tilde{h}$. Finally, by taking the limit as $N_1\to\infty$ and then as $\tilde{h}\to+0$ and by using a theorem given by Lebowitz and Martin-Löf (1972), we arrive at inequality (2.12).

3. Random-bond n-vector model

We consider the n-vector model (Stanley 1974) on a finite set Λ of N lattice sites: $N=|\Lambda|$. We consider two possibilities for the distribution of the exchange interaction.

In the first case, we assume that the system is described by the Hamiltonian

$$H = -\sum_{(i,j)} \sum_{\nu=1}^{n} J_{ij}^{(\nu)} s_i^{(\nu)} s_j^{(\nu)} - h \sum_i \sum_{\nu=1}^{n} \mu_i^{(\nu)} s_i^{(\nu)} \tag{3.1}$$

where $s_i^{(\nu)}$ is the νth component of the n-dimensional classical vector spin of unit magnitude for the site i ($i \in \Lambda$): $\Sigma_{\nu=1}^{n} (s_i^{(\nu)})^2 = 1$. $J_{ij}^{(\nu)}$ for the pair of sites (i, j) is a quenched random variable whose probability distribution is denoted by $\tilde{P}_{ij}^{(\nu)}(J_{ij}^{(\nu)})$, and is independent not only of $J_{kl}^{(\lambda)}$ for the other pairs of sites (k, l) and any λ but also of $J_{ij}^{(\lambda)}$ for the same pair of sites (i, j) and different superscript λ. h is the magnitude of the external field, and $\mu_i^{(\nu)}$ is the magnetic moment of the νth component of the spin on the site i, multiplied by the direction cosine of the external field in the ν direction. In the second case, we assume only one variable J_{ij} for the exchange interaction between a pair of sites (i, j), and the Hamiltonian is given by

$$H = -\sum_{(i,j)} J_{ij} \sum_{\nu=1}^{n} s_i^{(\nu)} s_j^{(\nu)} - h \sum_i \sum_{\nu=1}^{n} \mu_i^{(\nu)} s_i^{(\nu)} \tag{3.2}$$

where J_{ij} is independent of J_{kl} for the other pairs of sites (k, l) and its probability distribution is denoted by $\tilde{P}_{ij}(J_{ij})$.

For the n-vector model, we use the notation s_A^{ρ} for the following product of the spin variables on the sites in the finite set A of sites in the system

$$s_A^{\rho} \equiv \prod_{\substack{k \\ (k \in A)}} \prod_{\nu=1}^{n} (s_k^{(\nu)})^{\rho_\nu(k)} \tag{3.3}$$

where $\rho_\nu(k)$ are non-negative integers and we assume that $\Sigma_{\nu=1}^{n} \rho_\nu(k) \geq 1$. The canonical average of s_A^{ρ} is defined by

$$\langle s_A^{\rho} \rangle_{N,h,B_0}^{\{\beta J_{ij}^{(\nu)}\}} = \mathrm{Tr}\, \mathrm{e}^{-\beta H} s_A^{\rho} / \mathrm{Tr}\, \mathrm{e}^{-\beta H} \tag{3.4}$$

for the first case where H is given by (3.1), and by this equation with $\{\beta J_{ij}\}$ in the place of $\{\beta J_{ij}^{(\nu)}\}$ on the left-hand side for the second case where H is given by (3.2). Here Tr denotes

$$\mathrm{Tr} \ldots = \int \ldots \int \left(\prod_k \prod_{\nu=1}^{n} \mathrm{d}s_k^{(\nu)} \right) \ldots .$$

The integrations on the right-hand side are taken under the conditions $\{\Sigma_{\nu=1}^{n} (s_i^{(\nu)})^2 = 1\}$.

In the first case where the Hamiltonian is (3.1), we encounter n Ising models of spin ± 1, each of which consists of N_1 Ising spins of ± 1 on the set of sites Λ_1 where $A \subset \Lambda_1 \subset \Lambda$ and $N_1 = |\Lambda_1|$. The Hamiltonian for the νth one is given by

$$H_{\mathrm{I}}^{(\nu)} = \sum_{\substack{(i,j) \\ (i,j \in \Lambda_1)}} J_{ij}^{(\nu)} \sigma_i^{(\nu)} \sigma_j^{(\nu)} - \tilde{h}\tilde{\mu} \sum_{\substack{i \\ (i \in \Lambda_1)}} \sigma_i^{(\nu)} - \sum_{\substack{(i,j) \\ (i \in \Lambda_1, j \in \Lambda \setminus \Lambda_1)}} J_{ij}^{(\nu)} \sigma_i^{(\nu)} \tag{3.5}$$

where $\sigma_i^{(\nu)}$ is the spin variable for the site i, and $\tilde{h}$ and $\tilde{\mu}$ are positive. We later need the following spin correlation function

$$\langle \sigma_A^{(\nu)\rho_\nu} \rangle_{N,\tilde{h},B_1}^{\{\beta J_{ij}^{(\nu)}\}} \equiv \sum_{\{\sigma_i^{(\nu)}\}} \sigma_A^{(\nu)\rho_\nu} \exp(-\beta H_{\mathrm{I}}^{(\nu)}) \Big/ \sum_{\{\sigma_i^{(\nu)}\}} \exp(-\beta H_{\mathrm{I}}^{(\nu)}) \tag{3.6}$$

where

$$\sigma_A^{(\nu)\rho_\nu} \equiv \prod_{\substack{k \\ (k\in A)}} \sigma_k^{(\nu)\rho_\nu(k)}. \tag{3.7}$$

B_1 denotes the boundary condition that the spins $\sigma_j^{(\nu)}$ for $j \in \Lambda\backslash\Lambda_1$, which interact with a spin $\sigma_i^{(\nu)}$ for $i \in \Lambda_1$, are all plus one.

The configurational average of a function $Q\{J_{ij}^{(\nu)}\}$ of the set $\{J_{ij}^{(\nu)}\}$ is denoted by angular brackets with suffix c

$$\langle Q\{J_{ij}^{(\nu)}\}\rangle_c \equiv \int \ldots \int \left(\prod_{(i,j)} \prod_{\nu=1}^{n} \mathrm{d}J_{ij}^{(\nu)} \right) P\{J_{ij}^{(\nu)}\} Q\{J_{ij}^{(\nu)}\} \tag{3.8}$$

where

$$P\{J_{ij}^{(\nu)}\} \equiv \prod_{(i,j)} \prod_{\nu=1}^{n} \tilde{P}_{ij}^{(\nu)}(J_{ij}^{(\nu)}). \tag{3.9}$$

The thermodynamic limits of the correlation functions are defined in the same way as in § 2, namely,

$$|\langle\langle s_A^\rho \rangle^{\{\beta J_{ij}^{(\nu)}\}}\rangle_c| = \lim_{h\to+0} \lim_{N\to\infty} |\langle\langle s_A^\rho \rangle_{N,h,B_0}^{\{\beta J_{ij}^{(\nu)}\}}\rangle_c| \tag{3.10}$$

and for a fixed set $\{\gamma_{ij}^{(\nu)}\}$ of values $\gamma_{ij}^{(\nu)}$

$$\langle \sigma_A^{(\nu)\rho_\nu} \rangle_{(+)}^{\{\gamma_{ij}^{(\nu)}\}} = \lim_{h\to+0} \lim_{N\to\infty} \langle \sigma_A^{(\nu)\rho_\nu} \rangle_{N,h,B_0}^{\{\gamma_{ij}^{(\nu)}\}(+)} \tag{3.11}$$

and at $T=0$

$$|\langle\langle s_A^\rho \rangle^{\{\beta J_{ij}^{(\nu)}=\infty\}}\rangle_c| = \lim_{\beta h\to+0} \lim_{N\to\infty} \lim_{\beta J_{ij}^{(\nu)}\to\infty} |\langle\langle s_A^\rho \rangle_{N,h,B_0}^{\{\beta J_{ij}^{(\nu)}\}}\rangle_c|. \tag{3.12}$$

In the second case, the Hamiltonian H_I given by (2.4) plays the role of (3.5), and the following spin correlation function appears

$$\langle \sigma_A^\rho \rangle_{N,h,B_1}^{\{\beta J_{ij}\}} \equiv \sum_{\{\sigma_i\}} \sigma_A^\rho \, \mathrm{e}^{-\beta H_\mathrm{I}} \Big/ \sum_{\{\sigma_i\}} \mathrm{e}^{-\beta H_\mathrm{I}} \tag{3.13}$$

where

$$\sigma_A^\rho = \prod_{\substack{k \\ (k\in A)}} \prod_{\nu=1}^{n} \sigma_k^{\rho_\nu(k)}. \tag{3.14}$$

The configurational average with respect to $\{J_{ij}\}$ is defined by (2.7) and the thermodynamic limits (2.9)–(2.11) are used.

We now have the following theorem 2.

Theorem 2. For β, either finite or infinite, we have the following inequalities. In the first case when the system is described by the Hamiltonian (3.1), we have

$$|\langle\langle s_A^\rho \rangle^{\{\beta J_{ij}^{(\nu)}\}}\rangle_c| \leq \prod_{\nu=1}^{n} \langle \sigma_A^{(\nu)\rho_\nu} \rangle_{(+)}^{\{\langle|\beta_{ij}^{(\nu)} J_{ij}^{(\nu)}|\rangle_c\}} \tag{3.15}$$

where

$$\langle|\beta_{ij}^{(\nu)} J_{ij}^{(\nu)}|\rangle_c = \tfrac{1}{2} \int_{-\infty}^{\infty} |\ln[\tilde{P}_{ij}^{(\nu)}(J_{ij}^{(\nu)})/\tilde{P}_{ij}^{(\nu)}(-J_{ij}^{(\nu)})]| \tilde{P}_{ij}^{(\nu)}(J_{ij}^{(\nu)}) \, \mathrm{d}J_{ij}^{(\nu)}. \tag{3.16}$$

In the second case when the system is described by the Hamiltonian (3.2), we have

$$|\langle\langle s_A^\rho\rangle^{\{\beta J_{ij}\}}\rangle_c| \leq \langle\sigma_A^\rho\rangle_{(+)}^{\{\langle|\beta_{ij}J_{ij}|\rangle_c\}} \tag{3.17}$$

where

$$\langle|\beta_{ij}J_{ij}|\rangle_c = \tfrac{1}{2}\int_{-\infty}^{\infty} |\ln[\tilde{P}(J_{ij})/\tilde{P}(-J_{ij})]|\tilde{P}(J_{ij})\, dJ_{ij}. \tag{3.18}$$

Proof. The present proof is almost the same as the one for theorem 1 in § 2. In proving (3.15) with (3.16), we change the notations J_{ij}, $\tilde{P}_{ij}\{J_{ij}\}$, $\alpha_{ij}(|J_{ij}|)$, $\beta_{ij}(J_{ij})$, s_i and σ_i into $J_{ij}^{(\nu)}$, $\tilde{P}_{ij}^{(\nu)}(J_{ij}^{(\nu)})$, $\alpha_{ij}^{(\nu)}(|J_{ij}^{(\nu)}|)$, $\beta_{ij}^{(\nu)}(J_{ij}^{(\nu)})$, $s_i^{(\nu)}$ and $\sigma_i^{(\nu)}$, respectively, and have additional summations or products with respect to ν. We then have the following equations in place of equations (2.14)–(2.18)

$$\tilde{H} = -\sum_{\substack{(i,j)\\(i,i\in\Lambda)}}\sum_{\nu=1}^{n} J_{ij}^{(\nu)}s_i^{(\nu)}s_j^{(\nu)} - h\sum_{\substack{i\\(i\in\Lambda\backslash\Lambda_1)}}\sum_{\nu=1}^{n}\mu_i^{(\nu)}s_i^{(\nu)} \tag{3.19}$$

$$\langle\langle s_A^\rho\rangle_{N,(h,0),B_0}^{\{\beta J_{ij}^{(\nu)}\}}\rangle_c = \int\cdots\int\left(\prod_{(i,j)}\prod_{\nu=1}^{n} dJ_{ij}^{(\nu)}\right)\left(\prod_{(i,j)}\prod_{\nu=1}^{n}\alpha_{ij}^{(\nu)}\right)\exp\left(\sum_{(i,j)}\sum_{\nu=1}^{n}\beta_{ij}^{(\nu)}J_{ij}^{(\nu)}\right)\langle s_A^\rho\rangle_{N,(h,0),B_0}^{\{\beta J_{ij}^{(\nu)}\}} \tag{3.20}$$

$$\alpha_{ij}^{(\nu)} \equiv \alpha_{ij}^{(\nu)}(|J_{ij}^{(\nu)}|) = [\tilde{P}_{ij}^{(\nu)}(J_{ij}^{(\nu)})\tilde{P}_{ij}^{(\nu)}(-J_{ij}^{(\nu)})]^{1/2} \tag{3.21}$$

$$\beta_{ij}^{(\nu)} \equiv \beta_{ij}^{(\nu)}(|J_{ij}^{(\nu)}|) = \begin{cases}(1/2|J_{ij}^{(\nu)}|)\ln[\tilde{P}_{ij}^{(\nu)}(|J_{ij}^{(\nu)}|)/\tilde{P}_{ij}^{(\nu)}(-|J_{ij}^{(\nu)}|)] & J_{ij}^{(\nu)}\neq 0\\ 0 & J_{ij}^{(\nu)} = 0\end{cases} \tag{3.22}$$

$$\langle\langle s_A^\rho\rangle_{N,(h,0),B_0}^{\{\beta J_{ij}^{(\nu)}\}}\rangle_c = \int\cdots\int\left(\prod_{(i,j)}\prod_{\nu=1}^{n} dJ_{ij}^{(\nu)}\right)\left(\prod_{(i,j)}\prod_{\nu=1}^{n}\alpha_{ij}^{(\nu)}\right)$$
$$\times\exp\left(\sum_{(i,j)}\sum_{\nu=1}^{n}\beta_{ij}^{(\nu)}J_{ij}^{(\nu)}\sigma_i^{(\nu)}\sigma_j^{(\nu)}\right)\prod_{\nu=1}^{n}\sigma_A^{(\nu)\rho_\nu}\langle s_A^\rho\rangle_{N,(h,0),B_0}^{\{\beta J_{ij}^{(\nu)}\}} \tag{3.23}$$

where $\sigma_i^{(\nu)}$ is either +1 or −1 for $i\in\Lambda_1$ and +1 for $i\in\Lambda\backslash\Lambda_1$. By multiplying $\exp(\tilde{\beta}\tilde{h}\tilde{\mu}\sum_{i(i\in\Lambda_1)}\sum_{\nu=1}^{n}\sigma_i^{(\nu)})$ on both sides of (3.23), taking summation with respect to $\{\sigma_i^{(\nu)}\}$ over all the possible 2^{nN_1} sets of values of $\{\sigma_i^{(\nu)}\}$ and dividing by $[2\cosh(\tilde{\beta}\tilde{h}\tilde{\mu})]^{nN_1}$, we have

$$\langle\langle s_A^\rho\rangle_{N,(h,0),B_0}^{\{\beta J_{ij}^{(\nu)}\}}\rangle_c = \int\cdots\int\left(\prod_{(i,j)}\prod_{\nu=1}^{n} dJ_{ij}^{(\nu)}\right)P\{J_{ij}^{(\nu)}\}\prod_{\nu=1}^{n}\langle\sigma_A^{(\nu)\rho_\nu}\rangle_{N_1,\tilde{h},B_1}^{\{\beta_{ij}^{(\nu)}J_{ij}^{(\nu)}\}}\langle s_A^\rho\rangle_{N,(h,0),B_0}^{\{\beta J_{ij}^{(\nu)}\}} \tag{3.24}$$

where

$$P\{J_{ij}^{(\nu)}\} = \left(\prod_{(i,j)}\prod_{\nu=1}^{n}\alpha_{ij}^{(\nu)}\right)\left[\sum_{\{\sigma_i^{(\nu)}\}}\prod_{\nu=1}^{n}\exp\left(\sum_{(i,j)}J_{ij}^{(\nu)}\sigma_i^{(\nu)}\sigma_j^{(\nu)} + \tilde{\beta}\tilde{h}\tilde{\mu}\sum_{i\in\Lambda_1}\sigma_i^{(\nu)}\right)\right]$$
$$\times\{[2\cosh(\tilde{\beta}\tilde{h}\tilde{\mu})]^{nN_1}\}^{-1} \tag{3.25}$$

and $\langle\sigma_A^{(\nu)\rho_\nu}\rangle_{N_1,\tilde{h},B_1}^{\{\beta_{ij}^{(\nu)}J_{ij}^{(\nu)}\}}$ is defined by (3.6). The absolute value of the last factor in (3.24) is now overestimated by unity and then we have

$$|\langle\langle s_A^\rho\rangle_{N,(h,0),B_0}^{\{\beta J_{ij}^{(\nu)}\}}\rangle_c| \leqslant \prod_{\nu=1}^{n} \langle|\langle\sigma_A^{(\nu)\rho_\nu}\rangle_{N_1,\tilde{h},B_1}^{\{\beta_{ij}^{(\nu)}J_{ij}^{(\nu)}\}}|\rangle_c$$

$$\leqslant \prod_{\nu=1}^{n} \langle\sigma_A^{(\nu)\rho_\nu}\rangle_{N_1,\tilde{h},B_1}^{\{\langle|\beta_{ij}^{(\nu)}J_{ij}^{(\nu)}|\rangle_c\}} \tag{3.26}$$

instead of (2.21). By the same procedure as taken in the proof of theorem 1 for the thermodynamic limit, we have equation (3.15) with (3.16).

In proving (3.17) with (3.18), we change s_i into $s_i^{(\nu)}$ and have additional summations or products with respect to ν, in the proof of theorem 1. In place of (2.14), we have (3.19) where $J_{ij}^{(\nu)} = J_{ij}$ ($\nu = 1, 2, \ldots, n$). In (2.21) and (2.22), $\langle s_A^\rho\rangle_{N,(h,0),B_0}^{\{\beta J_{ij}^M\}(+)}$ and $\langle s_A^\rho\rangle_{(+)}^{\{\beta J_{ij}^M\}}$ are replaced by unity.

4. Disappearance of the spontaneous magnetisation

In the present section, we focus our attention on the averaged spontaneous magnetisation only for the systems of the nearest-neighbour interactions. The system of the Ising model with general spin S is assumed to be described by

$$H_S = -\sum_{\substack{(i,j)\\(i,j:NN)}} J_{ij}s_is_j - h\sum_i \mu_i s_i \tag{4.1}$$

where s_i takes on $-S, -S+1, \ldots, S$. The system of the n-vector model is assumed to be described by

$$H_n = -\sum_{\substack{(i,j)\\(i,j:NN)}} J_{ij}\mathbf{s}_i\cdot\mathbf{s}_j - h\sum_i\sum_{\nu=1}^{n} \mu_i^{(\nu)} s_i^{(\nu)} \tag{4.2}$$

where $\mathbf{s}_i$ is the n-dimensional classical spin of unit magnitude for the site i and $s_i^{(\nu)}$ is its νth component:

$$\mathbf{s}_i = (s_i^{(1)}, s_i^{(2)}, \ldots, s_i^{(n)}) \qquad |\mathbf{s}_i| = 1. \tag{4.3}$$

In both (4.1) and (4.2), J_{ij} for each nearest-neighbour pair of sites i and j is a quenched random variable whose probability distribution is denoted by $\tilde{P}(J_{ij})$ and assumed to be independent of J_{kl} for the other nearest-neighbour pairs of sites k and l.

We have now from theorem 1 for the Ising model with general spin S

$$|\langle\langle s_i\rangle^{\{\beta J_{ij}\}}\rangle_c| \leqslant |S| m_I(\langle|\beta_{ij}J_{ij}|\rangle_c) \tag{4.4}$$

and from theorem 2, especially from equation (3.17), for the n-vector model

$$|\langle\langle s_i^{(\lambda)}\rangle^{\{\beta J_{ij}\}}\rangle_c| \leqslant m_I(\langle|\beta_{ij}J_{ij}|\rangle_c) \tag{4.5}$$

where

$$\langle|\beta_{ij}J_{ij}|\rangle_c = \tfrac{1}{2}\int_{-\infty}^{\infty} |\ln[\tilde{P}(J_{ij})/\tilde{P}(-J_{ij})]|\tilde{P}(J_{ij})\, dJ_{ij}. \tag{4.6}$$

$m_I(\beta J)$ is the spontaneous magnetisation for the ferromagnetic Ising model of spin ± 1.

We define the spontaneous magnetisation in our systems by choosing $\mu_i = \mu > 0$ and $\mu_i^{(1)} = \mu > 0$, $\mu_i^{(\nu)} = 0$ $(\nu = 2, 3, \ldots, n)$, as follows

$$\bar{m}_S = \lim_{h\to+0} \lim_{N\to\infty} \langle\langle s_i \rangle_{N,h,B_0}^{\{\beta J_{ij}\}}\rangle_c \tag{4.7}$$

$$\bar{m}_n = \lim_{h\to+0} \lim_{N\to\infty} \langle\langle s_i^{(1)} \rangle_{N,h,B_0}^{\{\beta J_{ij}\}}\rangle_c \tag{4.8}$$

for the respective systems. As far as $|\langle\langle s_i\rangle^{\{\beta J_{ij}\}}\rangle_c| = 0$ or $|\langle\langle s_i^{(\lambda)}\rangle^{\{\beta J_{ij}\}}\rangle_c| = 0$, we have $\bar{m}_S = 0$ or $\bar{m}_n = 0$, accordingly. Thus we conclude that there is no spontaneous magnetisation in our systems in which the probability distribution of the exchange integrals satisfies the condition

$$\langle|\beta_{ij}J_{ij}|\rangle_c \leqslant J/k_B T_C \tag{4.9}$$

where T_C is the Curie temperature of the ferromagnetic Ising model of spin ± 1 with the exchange integral $J > 0$. Similar conclusions and the same condition (4.9) are obtained for the spontaneous long-range order parameter in any antiferromagnetic phase by suitably choosing the signs of μ_i or the values of $\mu_i^{(\nu)}$.

We now investigate the condition (4.9) for several special types of the probability distribution of J_{ij}.

4.1. *Discrete distribution of the three delta functions*

We assume that J_{ij} takes $J_0 > 0$, 0 and $-J_0$ with probabilities p, r and q, where $p + q + r = 1$. The probability distribution is formally expressed by

$$\tilde{P}(J_{ij}) = p\delta(J_{ij} - J_0) + r\delta(J_{ij}) + q\delta(J_{ij} + J_0). \tag{4.10}$$

In this case, the mean $\bar{J}$ and the standard deviation σ are given by $(2p + r - 1)J_0$ and $[1 - r - (2p - 1 + r)^2]^{1/2} J_0$, respectively. The left-hand side of (4.9) is calculated as

$$\langle|\beta_{ij}J_{ij}|\rangle_c = \tfrac{1}{2}(1 - r)\ln[p/(1 - r - p)]. \tag{4.11}$$

This expression agrees with the one obtained previously (Horiguchi and Morita 1981). By using $x \equiv \bar{J}/\sigma$, (4.11) is expressed as

$$\langle|\beta_{ij}J_{ij}|\rangle_c = \tfrac{1}{2}(1 - r)\ln\left(\frac{[(1 - r)(1 + x^2)]^{1/2} + x}{[(1 - r)(1 + x^2)]^{1/2} - x}\right) \qquad 0 \leqslant x < (r^{-1} - 1)^{1/2}. \tag{4.12}$$

This expression as a function of $x = \bar{J}/\sigma$ is shown by the chain curve and the double chain curve in figure 1 for $x \geqslant 0$, by setting $r = 0$ and $r = 0.5$, respectively. The critical values of x satisfying the equality in (4.9) are given in table 1 for several lattices. These are the exact lower bounds to the critical values of $\bar{J}/\sigma$ for the ferromagnetic state. The lower bounds of the critical concentrations of the ferromagnetic bonds are obtainable through the relation

$$p = \frac{1}{2}\left(\frac{x(1 - r)^{1/2}}{(1 + x^2)^{1/2}} + 1 - r\right). \tag{4.13}$$

The values for $r = 0$ are found in table 1 of the paper by Horiguchi and Morita (1981).

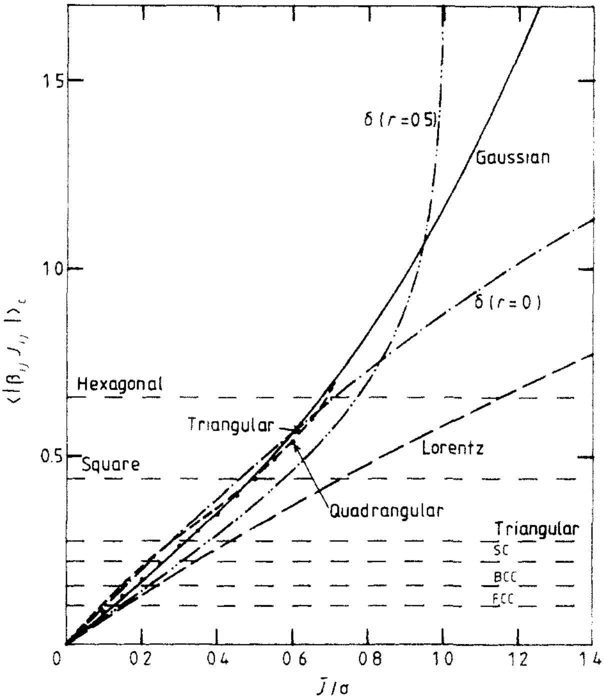

Figure 1. The graph of $\langle|\beta_{ij}J_{ij}|\rangle_c$ given by (4.6) for several types of probability distribution of J_{ij}. The chain curve is for the discrete distribution of three delta functions with $r = 0$ and the double chain curve for $r = 0.5$. The full curve is for the Gaussian distribution, the short-dashed curve for the triangular distribution and the long-dashed curve for the Lorentzian distribution. The dots are for the quadrangular distribution. The right-hand side of (4.9) is shown by horizontal dashed lines for the hexagonal, square, triangular, SC, BCC and FCC lattices.

Table 1. The lower bound to the critical value of $\bar{J}/\sigma$ for the ferromagnetic state. $\bar{J}$ is the mean and σ is the standard deviation for the discrete distribution of the three delta functions (δ, $r = 0$ and δ, $r = 0.5$), the Gaussian distribution (G), the quadrangular distribution (Q) and the triangular distribution (T). $\bar{J}$ is the median and σ is the width for the Lorentzian distribution (L).

	δ, $r = 0$	δ, $r = 0.5$	G	Q	T	L
Hexagonal	0.707 11	0.774 60	0.676 27	0.691 15	0.692 10	1.155 02
Square	0.455 09	0.577 35	0.493 44	0.500 46	0.494 70	0.728 70
Triangular	0.278 12	0.377 96	0.326 91	0.316 39	0.287 57	0.440 31
SC	0.223 51	0.308 14	0.268 26	0.255 74	0.223 07	0.352 91
BCC	0.158 05	0.220 72	0.193 66	0.181 71	0.149 80	0.248 92
FCC	0.102 27	0.143 87	0.126 93	0.117 88	0.092 22	0.160 83

4.2. Gaussian distribution

In this case, $\tilde{P}(J_{ij})$ is given by

$$\tilde{P}(J_{ij}) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(J_{ij} - \bar{J})^2\right) \tag{4.14}$$

and the left-hand side of (4.9) is expressed as

$$\langle|\beta_{ij}J_{ij}|\rangle_c = \sqrt{2/\pi}x\exp(-\tfrac{1}{2}x^2) + x^2\,\mathrm{erf}(x/\sqrt{2}) \tag{4.15}$$

where $\bar{J}$ is the mean and σ is the standard deviation and $x = \bar{J}/\sigma$. erf(x) is the error function (Magnus *et al* 1966). The graph of (4.15) for $x \geqslant 0$ is given by the full curve in figure 1 and the critical values of x are given in table 1.

4.3. Quadrangular distribution

We consider $\tilde{P}(J_{ij})$ given by

$$\tilde{P}(J_{ij}) = \begin{cases} 1/2b + aJ_{ij} & |J_{ij}| \leqslant b \\ 0 & \text{otherwise.} \end{cases} \tag{4.16}$$

The mean $\bar{J}$ and the standard deviation σ are given by $2ab^3/3$ and $(\frac{1}{3}b^2 - \frac{4}{9}a^2b^6)^{1/2}$, respectively. The left-hand side of (4.9) is expressed as

$$\begin{aligned}\langle|\beta_{ij}J_{ij}|\rangle_c &= \frac{1}{4ab^2}\ln(1-4a^2b^4) + \frac{1}{2}\ln\frac{1+2ab^2}{1-2ab^2} \\ &= \frac{(1+x^2)^{1/2}}{2\sqrt{3}x}\ln\frac{1-2x^2}{1+x^2} + \frac{1}{2}\ln\frac{(1+x^2)^{1/2}+\sqrt{3}x}{(1+x^2)^{1/2}-\sqrt{3}x}\end{aligned} \tag{4.17}$$

where $x = \bar{J}/\sigma$ and $|x| \leqslant 1/\sqrt{2}$. The graph of (4.17) for $x \geqslant 0$ is given by dots in figure 1. The maximum value of $\langle|\beta_{ij}J_{ij}|\rangle_c$ is $\ln 2$ at $x = 1/\sqrt{2}$ and very close to $J/k_BT_C = 0.6584789$ for the hexagonal lattice. Since our calculation gives the lower bound to the critical value of x for the ferromagnetic state, it might be possible that there is no ferromagnetic or antiferromagnetic state in the system with the probability distribution (4.16) on the hexagonal lattice. The critical values of x are given in table 1.

4.4. Triangular distribution

We consider $\tilde{P}(J_{ij})$ given by

$$\tilde{P}(J_{ij}) = \begin{cases} (b+J_{ij})/b(b+a) & -b \leqslant J_{ij} \leqslant a \\ (b-J_{ij})/b(b-a) & a \leqslant J_{ij} \leqslant b \\ 0 & \text{otherwise.} \end{cases} \tag{4.18}$$

The mean $\bar{J}$ and the standard deviation σ are given by $a/3$ and $[(a^2+3b^2)/18]^{1/2}$, respectively. The left-hand side of (4.9) is expressed as

$$\begin{aligned}\langle|\beta_{ij}J_{ij}|\rangle_c &= \frac{b}{b+a}\ln\left(1-\frac{a^2}{b^2}\right) + \frac{1}{2}\ln\frac{b+a}{b-a} \\ &= \frac{2-x^2-x(6-3x^2)^{1/2}}{2(1-2x^2)}\ln\frac{2(1-2x^2)}{2-x^2} + \frac{1}{2}\ln\frac{(6-3x^2)^{1/2}+3x}{(6-3x^2)^{1/2}-3x}\end{aligned} \tag{4.19}$$

where $x = \bar{J}/\sigma$ and $|x| \leqslant 1/\sqrt{2}$. The graph of (4.19) for $x \geqslant 0$ is given by the short-dashed line in figure 1. The maximum value of $\langle|\beta_{ij}J_{ij}|\rangle_c$ is $\ln 2$ at $x = 1/\sqrt{2}$. The situation for the hexagonal lattice is the same as in the quadrangular distribution. The critical values of x are given in table 1.

4.5. Lorentzian distribution

In this case, $\tilde{P}(J_{ij})$ is given by

$$\tilde{P}(J_{ij}) = \frac{\sigma}{\pi[(J_{ij}-\bar{J})^2+\sigma^2]} \tag{4.20}$$

and the left-hand side of (4.9) is expressed as

$$\langle|\beta_{ij}J_{ij}|\rangle_c = \frac{2}{\pi}\int_0^x \frac{1}{t}\tan^{-1} t\, dt \qquad x \geqslant 0 \tag{4.21}$$

where $\bar{J}$ is the median and σ is the width and $x = \bar{J}/\sigma$. The graph of (4.21) for $x \geqslant 0$ is given by the long-dashed line in figure 1 and the critical values of x are given in table 1.

5. Concluding remarks

We considered the random-bond Ising model of general spin S and the random-bond n-vector model, in both of which the exchange integrals are quenched random variables and their probability distributions are continuous. In both systems, we proved that the spin correlation functions in the thermodynamic limit of the zero external field limit are bounded above by non-trivial bounds. Applying the results to the spin on a single site, we found a sufficient condition for disappearance of the spontaneous long-range order parameter for these systems. The condition was examined for several types of the probability distribution.

Now we wish to mention an extension of the theorems obtained in the present paper to the configurational average of a product of spin correlation functions. For the Ising model of general spin S, we have

$$\left|\left\langle \prod_{l=1}^{L} \langle s_{A_l}^{\rho_l}\rangle^{\{\beta J_{ij}\}}\right\rangle_c\right| \leqslant S^*\left\langle \prod_{l=1}^{L} \sigma_{A_l}^{\rho_l}\right\rangle_{(+)}^{\{\langle|\beta_{ij}J_{ij}|\rangle_c\}} \tag{5.1}$$

where A_l are subsets of the set Λ of N lattice sites, and

$$s_{A_l}^{\rho_l} \equiv \prod_{\substack{k\\(k\in A_l)}} s_k^{\rho_l(k)} \qquad \sigma_{A_l}^{\rho_l} = \prod_{\substack{k\\(k\in A_l)}} \sigma_k^{\rho_l(k)}.$$

$\rho_l(k)$ are positive integers not greater than $2S$. S^* is equal to $|S|$ to the power of $\Sigma_{l=1}^{L} |A_l|^*$, where $|A_l|^* = \Sigma_{k(k\in A_l)} \rho_l(k)$. The thermodynamic limits

$$\left|\left\langle\left\langle \prod_{l=1}^{L} s_{A_l}^{\rho_l}\right\rangle^{\{\beta J_{ij}\}}\right\rangle_c\right| \qquad \text{and} \qquad \left\langle \prod_{l=1}^{L} \sigma_{A_l}^{\rho_l}\right\rangle_{(+)}^{\{\gamma_{ij}\}}$$

for a fixed set $\{\gamma_{ij}\}$ of values γ_{ij} are defined by similar equations to (2.9) and (2.10). For the n-vector model, we have

$$\left|\left\langle \prod_{l=1}^{L} \langle s_{A_l}^{\rho_l}\rangle^{\{\beta J_{ij}\}}\right\rangle_c\right| \leqslant \left\langle \prod_{l=1}^{L} \sigma_{A_l}^{\rho_l}\right\rangle_{(+)}^{\{\langle|\beta_{ij}J_{ij}|\rangle_c\}} \tag{5.2}$$

in place of (3.17). Here

$$s_{A_l}^{\rho_l} = \prod_{\substack{k\\(k\in A_l)}} \prod_{\nu=1}^{n} [s_k^{(\nu)}]^{\rho_{\nu,l}(k)} \qquad \sigma_{A_l}^{\rho_l} = \prod_{\substack{k\\(k\in A_l)}} \prod_{\nu=1}^{n} \sigma_k^{\rho_{\nu,l}(k)}.$$

$\rho_{\nu,l}(k)$ are non-negative integers and we assume that $\Sigma_{\nu=1}^{n}\rho_{\nu,l}(k)\geqslant 1$. The quantities appearing in both sides of equation (5.2) are defined in a similar way to (2.9) and (2.10). Extension of (3.15) is also possible, but we omit it here. Proofs of inequalities (5.1) and (5.2) are performed in the same ways as those for theorems 1 and 2. We do not reproduce them here.

We apply the above inequalities (5.1) and (5.2) to the configurational average of products $\langle s_i\rangle^{\{\beta J_{ij}\}}\langle s_j\rangle^{\{\beta J_{ij}\}}$ for the Ising model and $\langle s_i^{(\lambda)}\rangle^{\{\beta J_{ij}\}}\langle s_j^{(\lambda')}\rangle^{\{\beta J_{ij}\}}$ for the n-vector model discussed in § 4. We have, for $i\neq j$,

$$|\langle\langle s_i\rangle^{\{\beta J_{ij}\}}\langle s_j\rangle^{\{\beta J_{ij}\}}\rangle_{\rm c}|\leqslant|S|^2\langle\sigma_i\sigma_j\rangle_{(+)}^{(\langle|\beta_{ij}J_{ij}|\rangle_{\rm c})}\tag{5.3}$$

for the Ising model, and

$$|\langle\langle s_i^{(\lambda)}\rangle^{\{\beta J_{ij}\}}\langle s_j^{(\lambda')}\rangle^{\{\beta J_{ij}\}}\rangle_{\rm c}|\leqslant\langle\sigma_i\sigma_j\rangle_{(+)}^{(\langle|\beta_{ij}J_{ij}|\rangle_{\rm c})}\tag{5.4}$$

for the n-vector model. The right-hand sides of (5.3) and (5.4) are zero for the case of $\langle|\beta_{ij}J_{ij}|\rangle_{\rm c}=0$, which is possible at least for the probability distribution of J_{ij} satisfying the condition $\tilde{P}(J_{ij})=\tilde{P}(-J_{ij})$. Under the condition $\langle|\beta_{ij}J_{ij}|\rangle_{\rm c}=0$, we have

$$k_{\rm B}T\chi=\lim_{N\to\infty}\sum_j\{\langle\langle s_is_j\rangle^{\{\beta J_{ij}\}}\rangle_{\rm c}-\langle\langle s_i\rangle^{\{\beta J_{ij}\}}\langle s_j\rangle^{\{\beta J_{ij}\}}\rangle_{\rm c}\}=1-q\tag{5.5}$$

for the Ising model, and

$$k_{\rm B}T\chi=\lim_{N\to\infty}\sum_j\{\langle\langle s_i^{(\lambda)}s_j^{(\lambda)}\rangle^{\{\beta J_{ij}\}}\rangle_{\rm c}-\langle\langle s_i^{(\lambda)}\rangle^{\{\beta J_{ij}\}}\langle s_j^{(\lambda)}\rangle^{\{\beta J_{ij}\}}\rangle_{\rm c}\}$$
$$=1-q\tag{5.6}$$

for the n-vector model, where q is the Edwards–Anderson order parameter.

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